

mvsksPortfolios: portfolio tilting to harvest higher moment gains

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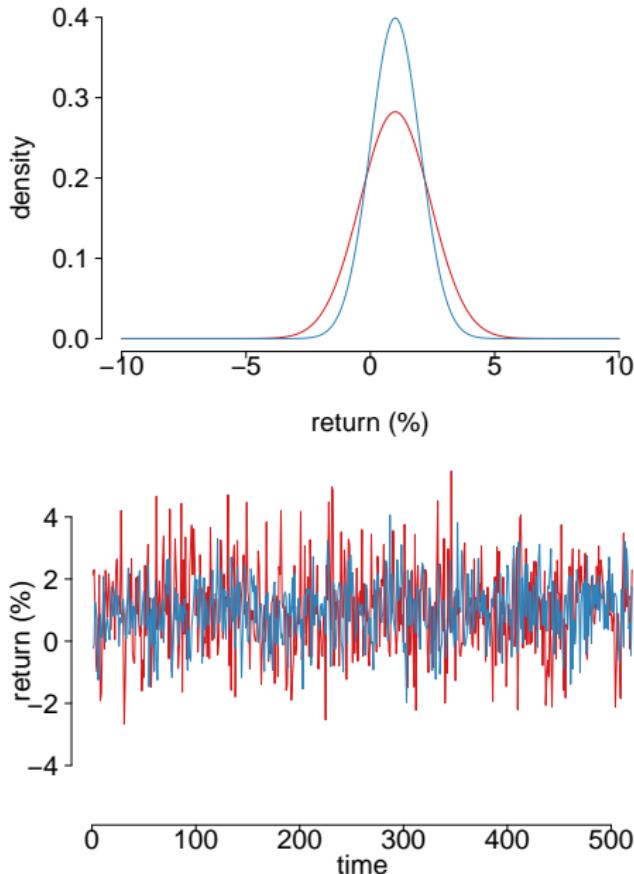
April 17, 2019

Moment preferences - volatility

Let \mathbf{X} be a p -dimensional random variable with finite fourth-order moments.

Let $\mathbf{w} \in \mathcal{C} \subset \mathbb{R}^p$ be the weights of a portfolio.

- Expected value: $\mu = \mathbb{E}[\mathbf{X}]$
- Portfolio expected value:
$$\mu_p = \mathbf{w}'\mu$$
- Covariance matrix:
$$\Sigma = \mathbb{E}[(\mathbf{X} - \mu)(\mathbf{X} - \mu)']$$
- Portfolio variance:
$$\sigma_p^2 = \mathbf{w}'\Sigma\mathbf{w}$$

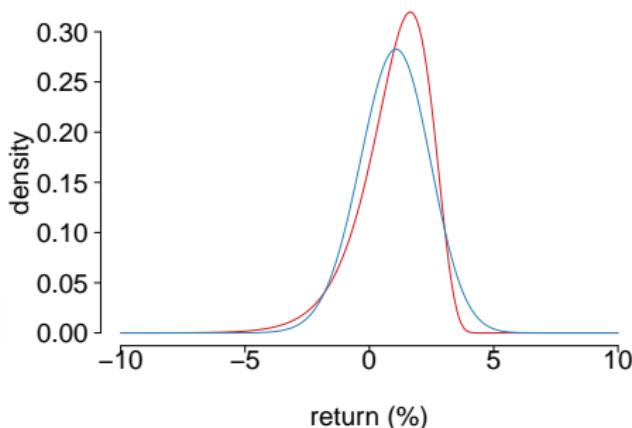


Moment preferences - skewness

The coskewness matrix is of size $p \times p^2$ and gathers all joint third-order central moments.

- Coskewness matrix:

$$\Phi = \mathbb{E}[(\mathbf{X} - \mu)(\mathbf{X} - \mu)' \otimes (\mathbf{X} - \mu)']$$

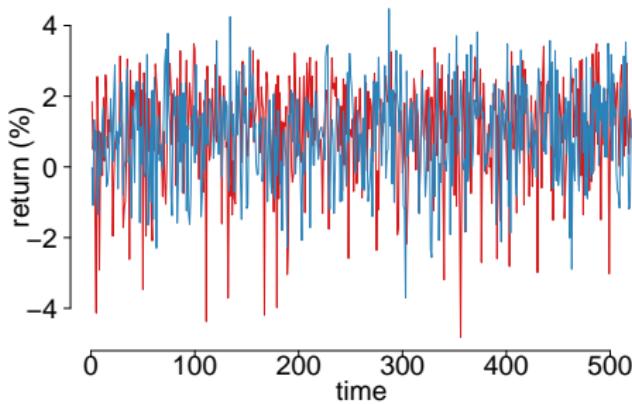


- Portfolio skewness:

$$\phi_p = \mathbf{w}' \Phi (\mathbf{w} \otimes \mathbf{w})$$

- Standardized skewness:

$$\text{skew} = \frac{\phi}{\sigma^3}$$

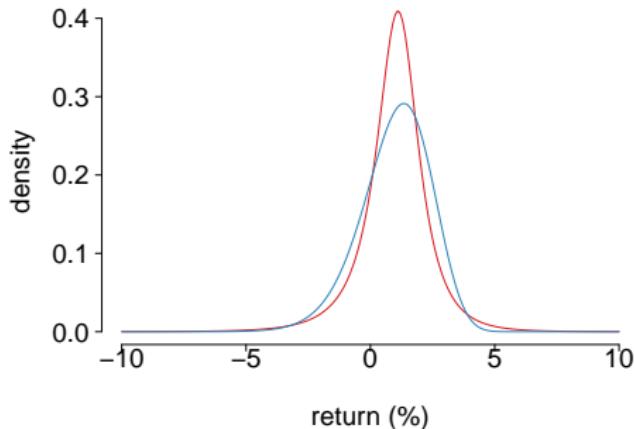


Moment preferences - kurtosis

The cokurtosis matrix is of size $p \times p^3$ and gathers all joint fourth-order central moments.

- Cokurtosis matrix:

$$\Psi = \mathbb{E}[(\mathbf{X} - \mu)(\mathbf{X} - \mu)' \otimes (\mathbf{X} - \mu)' \otimes (\mathbf{X} - \mu)']$$

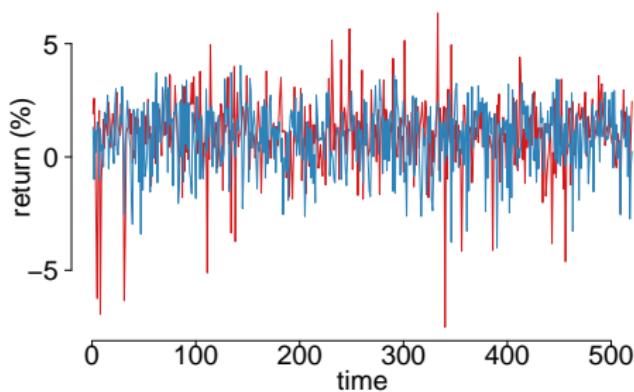


- Portfolio kurtosis:

$$\psi_p = \mathbf{w}' \Phi(\mathbf{w} \otimes \mathbf{w} \otimes \mathbf{w})$$

- Excess kurtosis:

$$\text{Ex. kurt.} = \frac{\psi}{\sigma^4} - 3$$



Moment preferences- other approaches

$$h(\mu_{\mathbf{w}' \mathbf{X}}, \sigma_{\mathbf{w}' \mathbf{X}}^2, \phi_{\mathbf{w}' \mathbf{X}}, \psi_{\mathbf{w}' \mathbf{X}})$$

- Expected utility:

$$\begin{aligned} \mathcal{U}_\gamma(\mathbf{w}) \approx & -\frac{\gamma}{2} \mathbf{w}' \widehat{\Sigma} \mathbf{w} + \frac{\gamma(\gamma+1)}{6} \mathbf{w}' \widehat{\Phi} (\mathbf{w} \otimes \mathbf{w}) \\ & - \frac{\gamma(\gamma+1)(\gamma+2)}{24} \mathbf{a}' \widehat{\Psi} (\mathbf{w} \otimes \mathbf{w} \otimes \mathbf{w}) \end{aligned}$$

- Cornish-Fisher Value-at-Risk:

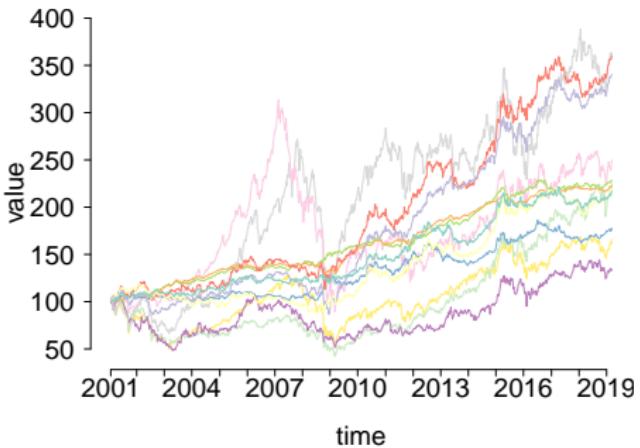
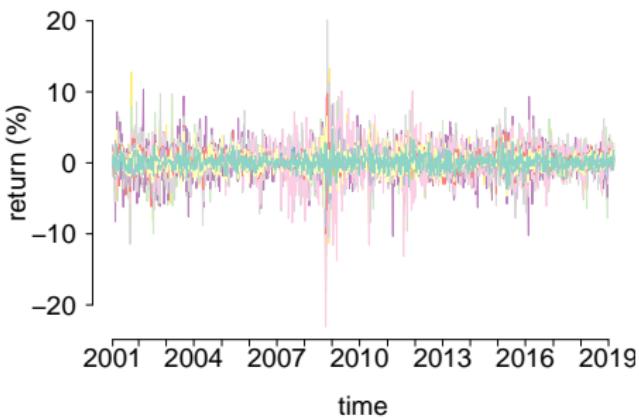
$$\text{VaR}_q(\mathbf{w}) \approx \widehat{\mu} + \widehat{\sigma} \left(z_q + \frac{\widehat{\phi}}{6} (z_q^2 - 1) + \frac{\widehat{\psi}}{24} (z_q^3 - 3z_q) - \frac{\widehat{\phi}^2}{36} (2z_q^3 - 5z_q) \right)$$

- Cornish-Fisher Tail Value-at-Risk

Data summary

	geom. mean	volatility	skewness	ex. kurt.
MSCI Europe	2.82	0.19	-0.71	7.46
MSCI USA	4.55	0.18	-0.34	3.05
MSCI Japan	1.63	0.19	-0.11	1.31
MSCI Emerging Markets	7.35	0.21	-0.15	4.97
EPRA Eurozone	5.16	0.19	-1.29	9.01
JPM EMU	4.64	0.04	-0.14	2.16
iBoxx EUR Corporate	4.50	0.03	-0.94	5.41
Bloomberg Barclays Global Aggregate Treasuries	3.17	0.06	0.57	4.01
JPM EMBI Global Diversified Composite	7.29	0.11	-0.12	4.87
Bloomberg Barclays Global High Yield	6.96	0.10	-0.21	4.70
Bloomberg Barclays US Corporates	4.37	0.10	0.29	1.46
Bloomberg Barclays World Govt Inflation Linked	4.34	0.07	0.11	1.39

Data summary



Optimized portfolios

Portfolios are always to solution to some objective function

- Equally-weighted portfolio (EW)

$$h_{\text{EW}}(\mathbf{w} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\zeta}) = \sum_{i=1}^N w_i^2$$

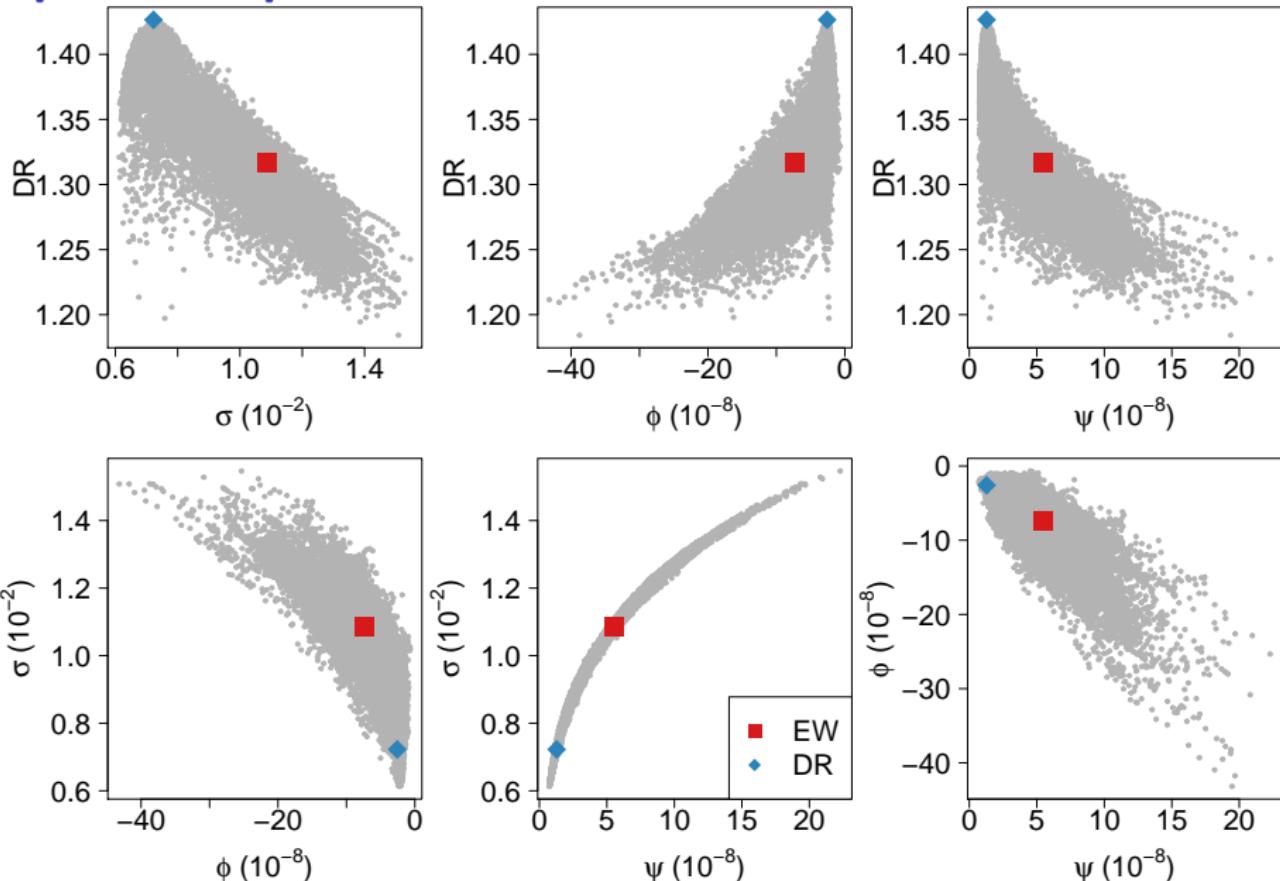
- Maximum diversification portfolio (DR) (Choueifaty & Coignard (2008), Journal of Portfolio Management)

$$h_{\text{DR}}(\mathbf{w} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\zeta}) = -\frac{\mathbf{w}' \sqrt{\text{diag}(\boldsymbol{\Sigma})}}{\sqrt{\mathbf{w}' \boldsymbol{\Sigma} \mathbf{w}}}$$

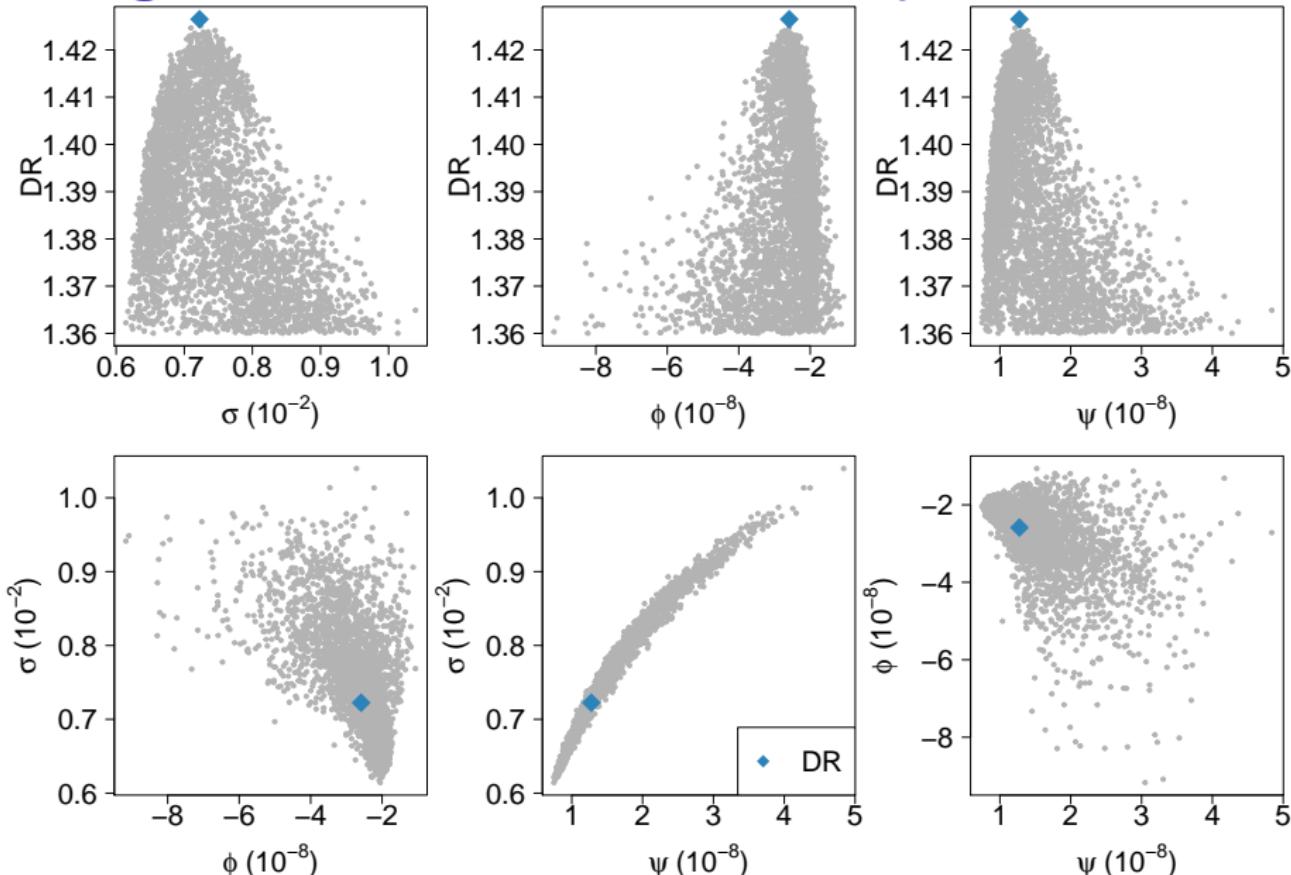
- Minimum tracking error volatility portfolio (TE)

$$h_{\text{TE}}(\mathbf{w} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\zeta}) = \sqrt{(\mathbf{w} - \mathbf{w}_0)' \boldsymbol{\Sigma} (\mathbf{w} - \mathbf{w}_0)}$$

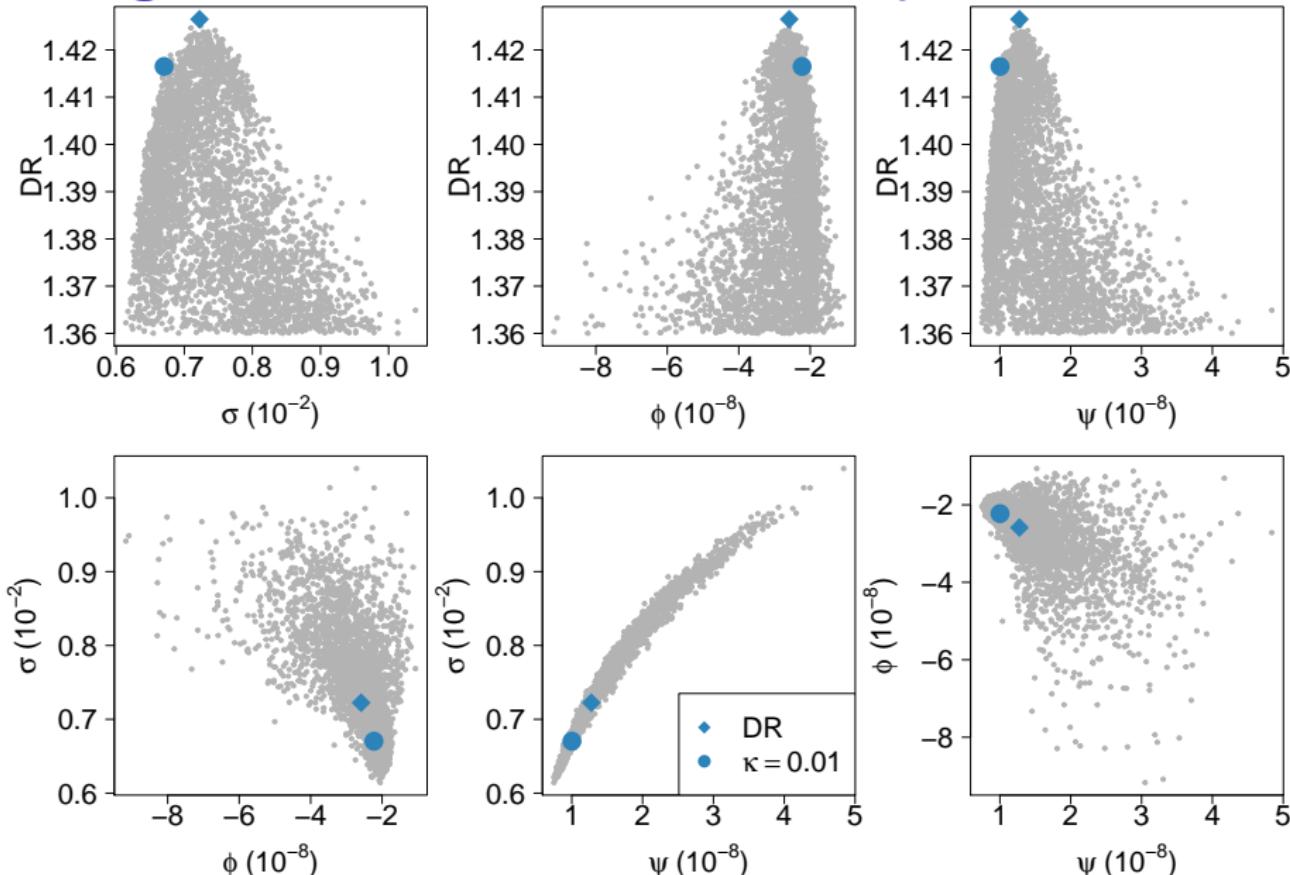
Optimized portfolios - characteristics



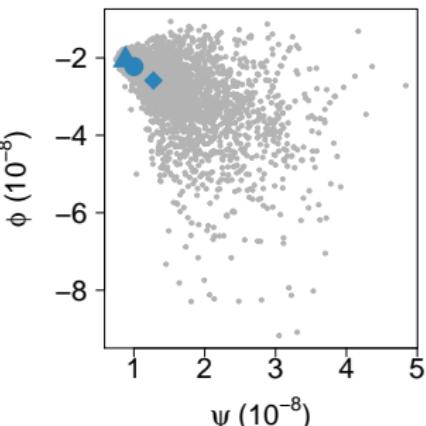
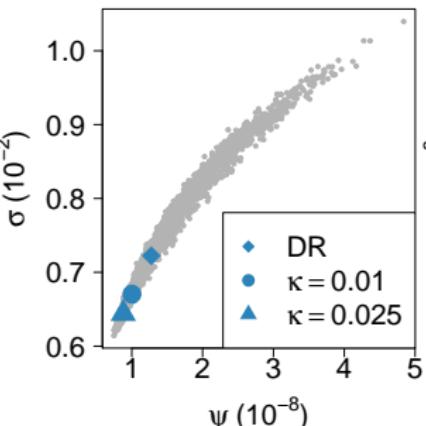
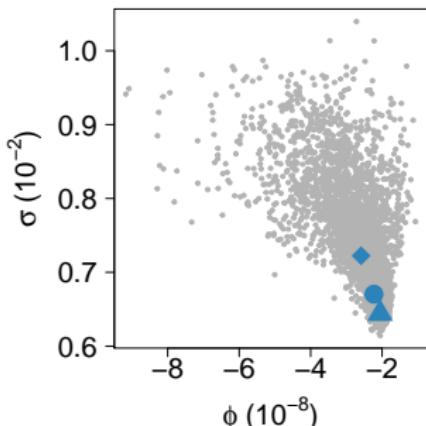
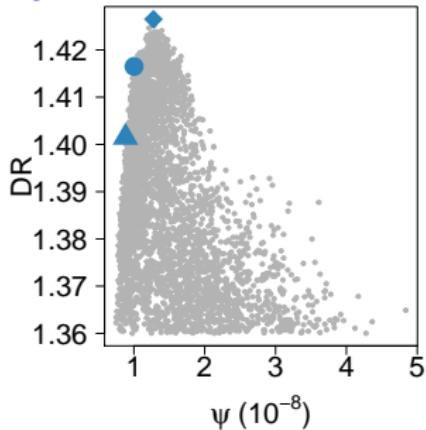
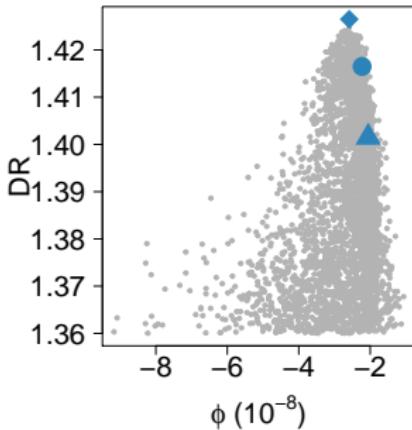
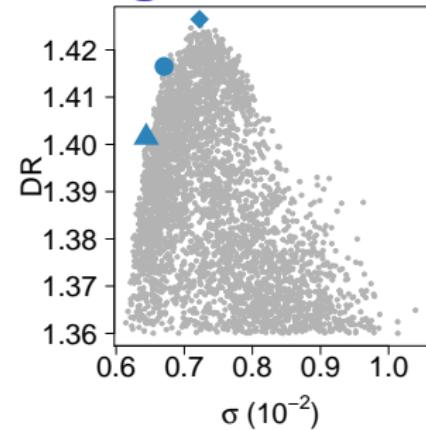
Tilting the maximum diversification portfolio



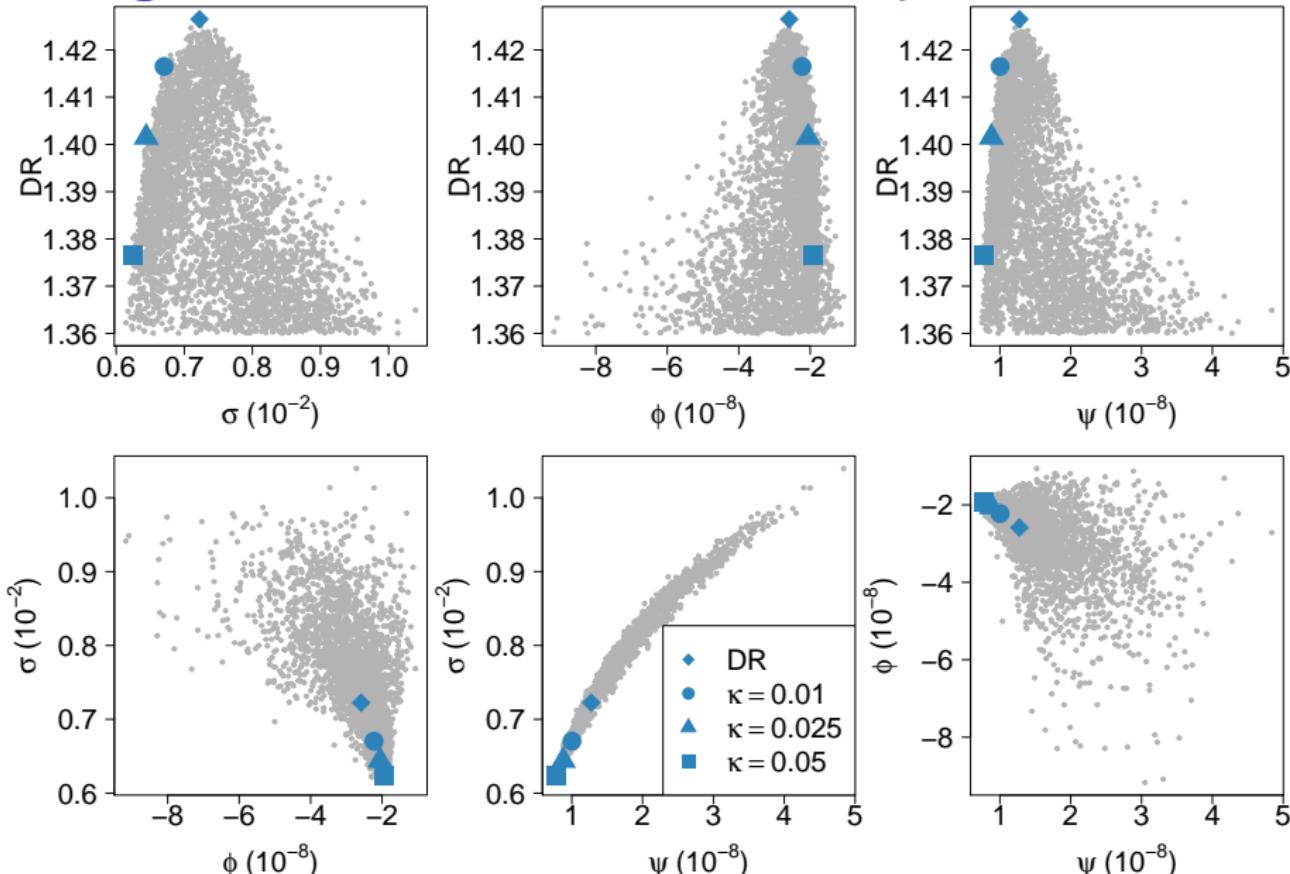
Tilting the maximum diversification portfolio



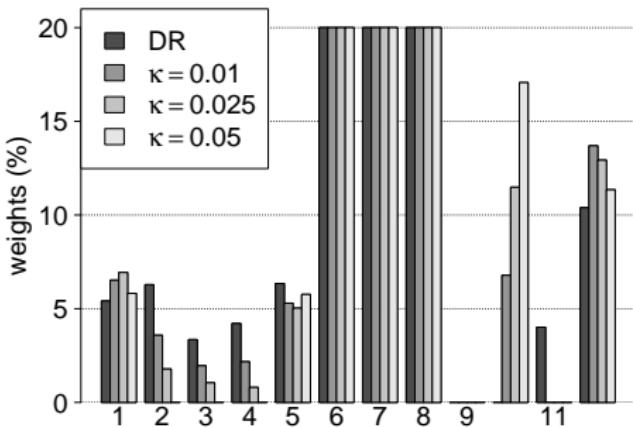
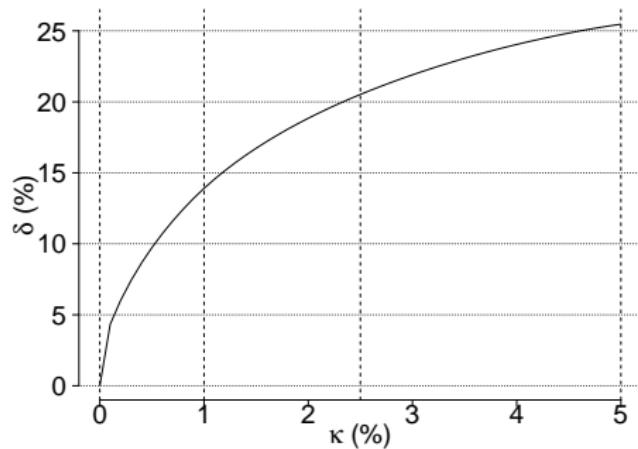
Tilting the maximum diversification portfolio



Tilting the maximum diversification portfolio



Tilting the maximum diversification portfolio



Mean-variance-skewness-kurtosis portfolio tilting

$$\underset{\delta \in \mathbb{R}, \mathbf{w} \in \mathcal{C}}{\text{maximize}} \quad \delta$$

subject to $h(\mathbf{w} \mid \mu, \Sigma, \Phi, \Psi, \zeta) \geq h(\mathbf{w}_0 \mid \mu, \Sigma, \Phi, \Psi, \zeta) - \kappa,$
 $\mathbf{w}'\mu \geq \mathbf{w}'_0\mu + \delta_\mu,$
 $\mathbf{w}'\Sigma\mathbf{w} \leq \mathbf{w}'_0\Sigma\mathbf{w}_0 - \delta_\Sigma,$
 $\mathbf{w}'\Phi(\mathbf{w} \otimes \mathbf{w}) \geq \mathbf{w}'_0\Phi(\mathbf{w}_0 \otimes \mathbf{w}_0) + \delta_\Phi,$
 $\mathbf{w}'\Psi(\mathbf{w} \otimes \mathbf{w} \otimes \mathbf{w}) \leq \mathbf{w}'_0\Psi(\mathbf{w}_0 \otimes \mathbf{w}_0 \otimes \mathbf{w}_0) - \delta_\Psi,$

where $(\delta_\mu, \delta_\Sigma, \delta_\Phi, \delta_\Psi)' = g(\delta)$ and κ determines the maximum deterioration in the initial objective function.

Reference: Boudt, Cornilly, Van Holle and Willems (2019). “Algorithmic portfolio tilting to harvest higher moment gains.”
<https://ssrn.com/abstract=3378491>.

Mean-variance-skewness-kurtosis portfolio tilting

$$\underset{\delta \in \mathbb{R}, \mathbf{w} \in \mathcal{C}}{\text{maximize}} \quad \delta$$

$$\text{subject to} \quad \text{TE}_{\text{vol}}(\mathbf{w}) \leq \tau,$$

$$\mathbf{w}'\boldsymbol{\mu} \geq \mathbf{w}'_0\boldsymbol{\mu} + \delta_\mu,$$

$$\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \leq \mathbf{w}'_0\boldsymbol{\Sigma}\mathbf{w}_0 - \delta_\Sigma,$$

$$\mathbf{w}'\boldsymbol{\Phi}(\mathbf{w} \otimes \mathbf{w}) \geq \mathbf{w}'_0\boldsymbol{\Phi}(\mathbf{w}_0 \otimes \mathbf{w}_0) + \delta_\Phi,$$

$$\mathbf{w}'\boldsymbol{\Psi}(\mathbf{w} \otimes \mathbf{w} \otimes \mathbf{w}) \leq \mathbf{w}'_0\boldsymbol{\Psi}(\mathbf{w}_0 \otimes \mathbf{w}_0 \otimes \mathbf{w}_0) - \delta_\Psi,$$

where $(\delta_\mu, \delta_\Sigma, \delta_\Phi, \delta_\Psi)' = g(\delta)$ and τ determines the maximum tracking error volatility of the tilted portfolio as compared to the reference portfolio.

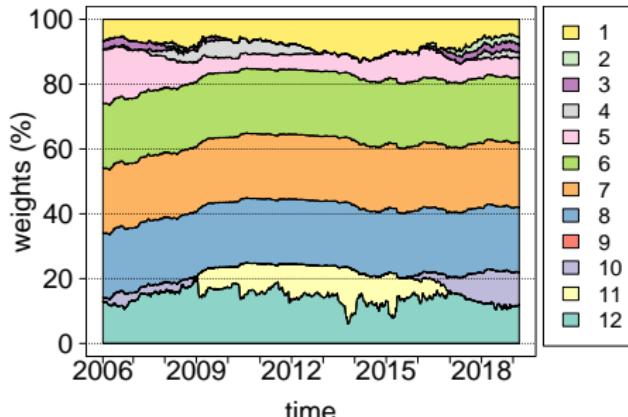
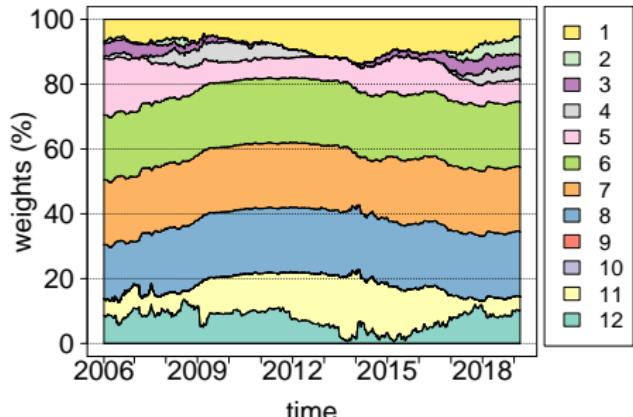
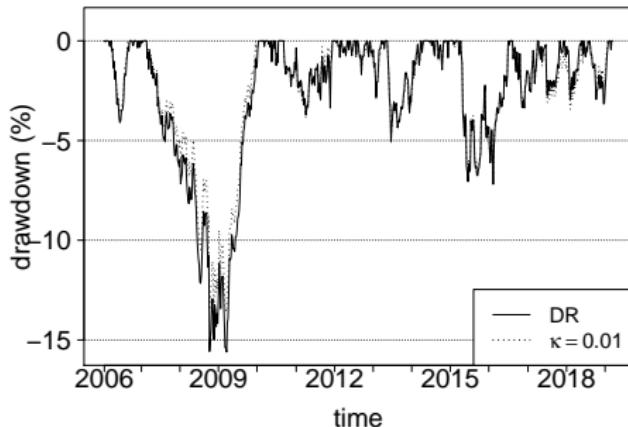
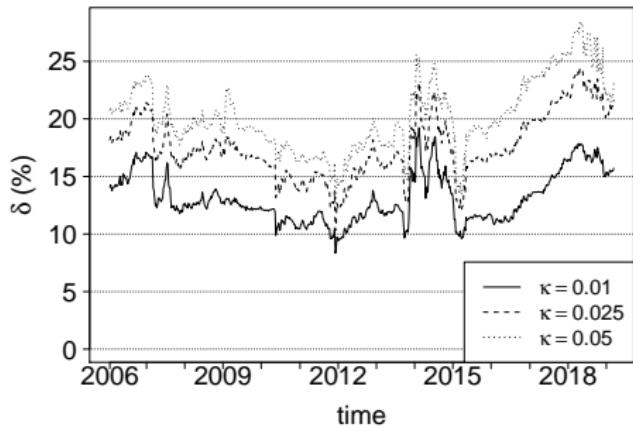
$$\text{TE}_{\text{vol}}(\mathbf{w}) = \sqrt{(\mathbf{w} - \mathbf{w}_0)' \boldsymbol{\Sigma} (\mathbf{w} - \mathbf{w}_0)}.$$

Out-of-sample tilting results

- Data: 12 indices from January 26, 2001 until March 15, 2019,
- Weekly rebalancing, five-year rolling estimation window,
- Full investment, no short-selling, maximum weight of 20%.

	geometric mean (%)	standard deviation (ann.)	m3 (1e-6)	m4 (1e-7)	max. drawdown (%)	ann. turnover (%)	ASSR
Maximum diversification							
DR	4.506	0.052	-0.222	0.228	-15.605	86.748	0.671
DR - tilted 1%	4.318	0.049	-0.197	0.190	-13.603	85.722	0.694
DR - tilted 2.5%	4.326	0.048	-0.171	0.166	-12.392	92.995	0.718
DR - tilted 5%	4.473	0.047	-0.147	0.145	-11.053	99.671	0.761
Equally-weighted							
EW	5.447	0.084	-1.305	2.007	-26.562	55.653	0.509
EW - tilted 1%	5.175	0.066	-0.559	0.713	-19.264	50.002	0.590
EW - tilted 2.5%	4.824	0.059	-0.333	0.402	-16.088	95.236	0.617
EW - tilted 5%	4.670	0.050	-0.197	0.211	-12.570	73.513	0.705

Out-of-sample tilting results



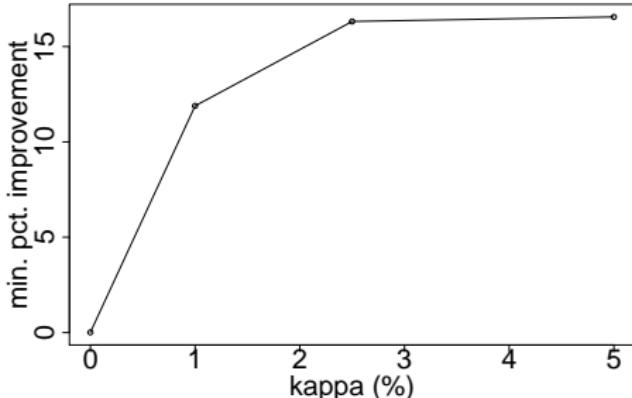
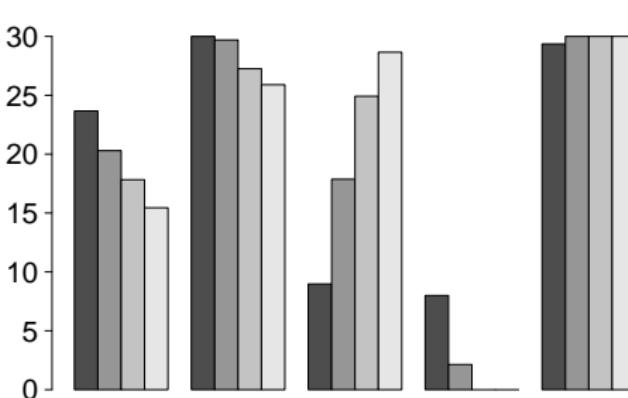
Code example

```
# load data
library(PerformanceAnalytics)
data(edhec)
x <- edhec[, 1:5]

# estimate moments
m1 <- colMeans(x)
M2 <- cov(x)
M3 <- M3.MM(x, as.mat = FALSE)
M4 <- M4.MM(x, as.mat = FALSE)
```

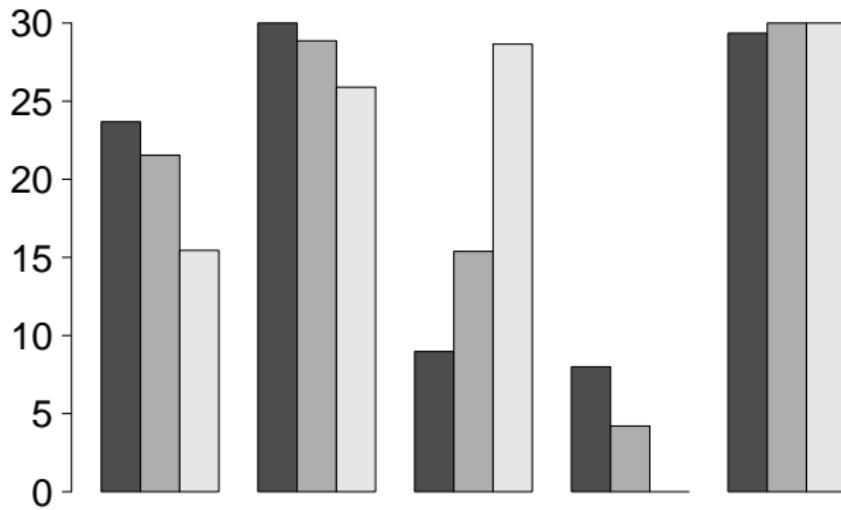
Code example

```
library(mvskPortfolios)
kappas <- c(0, 0.01, 0.025, 0.05)
# MVSK tilting - starting from "DR", tilting with margin on "DR"
resMVK <- mvskPortfolio(m1 = m1, M2 = M2, M3 = M3, M4 = M4,
                        w0 = "DR", g = "mvsk", ub = rep(0.3, 5),
                        href = "DR", kappa = kappas)
barplot(resMVK$w * 100, beside = TRUE, las = 1, cex.axis = 2.5)
par(mar = c(5.1, 5.1, 4.1, 2.1), cex.lab = 2.5, cex.axis = 2.5)
plot(kappas * 100, resMVK$delta * 100, type = 'o', lwd = 2,
     ylab = "min. pct. improvement", xlab = "kappa (%)")
```



Code example

```
# VSK tilting - starting from "DR", tilting with maximum  
# tracking error volatility  
resVSK <- mvsksPortfolio(M2 = M2, M3 = M3, M4 = M4, w0 = "DR",  
                           g = "mvsks", ub = rep(0.3, 5),  
                           href = "TEvol",  
                           kappa = c(0, 0.001, 0.025))  
barplot(resVSK$w * 100, beside = TRUE, las = 1, cex.axis = 2.5)
```



Main function

```
mvsksPortfolio(m1 = NULL, M2 = NULL, M3 = NULL, M4 = NULL,  
w0 = NULL, g = NULL, lb = NULL, ub = NULL,  
lin_eq = NULL, lin_eqC = NULL, nlin_eq = NULL,  
lin_ieq = NULL, lin_ieqC = NULL, nlin_ieq = NULL,  
href = NULL, kappa = NULL, relative = FALSE,  
param = NULL, options = list(), mompref = NULL)
```

Selected references

- Boudt, K., Cornilly, D., Van Holle & F., Willems, J. "Algorithmic portfolio tilting to harvest higher moment gains" (2019). <https://ssrn.com/abstract=3378491>.
- Boudt, K., Cornilly D. & Verdonck, T. "A Coskewness shrinkage approach for estimating the skewness of linear combinations of random variables." (2019). *Journal of Financial Econometrics* (forthcoming)
- Boudt, K., Cornilly, D. & Verdonck, T. "Nearest comoment estimation with unobserved factors" (2019). <https://ssrn.com/abstract=3087336>
- Briec, W., Kerstens, K., & Jokung, O. "Mean-variance-skewness portfolio performance gauging: a general shortage function and dual approach" (2007). *Management Science*, 53(1), 135–149.
- Choueifaty, Y. & Coignard, Y. (2008). "Toward maximum diversification" (2008). *Journal of Portfolio Management*, 35(1), 40–51.